Definitions

(a) Let f be a function f defined on the interval $I = (s, t)$. f is said to be differentiable at a point $x \in I$ if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists.}$$

(b) A function f is said to be differentiable on the interval $I = (s, t)$ if f is differentiable at each point in I.

Remark f is differentiable on an interval $I = (s, t)$ if and only if for each $x \in I$ there exists $l = l(x) \in \mathbb{R}$ such that the function

$$o(h) = f(x+h) - f(x) - l\,h \quad \text{for all } x + h \text{ in } I$$

satisfies that

$$\lim_{h \to 0} \frac{o(h)}{h} = 0.$$

Here, the function $o(h)$ is read as "little o of h" and $f'(x) = l(x) = l$ if l exists.

Proof

$\boxed{\Longrightarrow}$ Suppose that f is differentiable at some point $x \in I$, i.e.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists.}$$

By setting

$$l = f'(x) \text{ and } o(h) = f(x+h) - f(x) - l\,h \quad \text{for } x + h \in I,$$

we have

$$\lim_{h \to 0} \frac{o(h)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x) - l\,h}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - f'(x) = 0.$$

$\boxed{\Longleftarrow}$ Let $x \in I$. Suppose that $l = l(x)$ is a real number such that the function

$$o(h) = f(x+h) - f(x) - l\,h \quad \text{for all } x + h \text{ in } I$$

satisfies that

$$\lim_{h \to 0} \frac{o(h)}{h} = 0.$$

Then we have

$$0 = \lim_{h \to 0} \frac{o(h)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - l.$$

Hence,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = l \quad \text{exists.}$$

By setting $f'(x) = l$, we have shown that f is differentiable at $x \in I$.

Theorem (the Chain Rule) Let f be differentiable for all $x \in I = (s,t)$ and let g be differentiable for all $y \in J = (u,v)$. Suppose that

$$\text{Range of } f \;=\; f(I) \;=\; \{f(x) \mid x \in I\} \subseteq J \;=\; \text{Domain of } g.$$

Then $g \circ f$ is differentiable for all $x \in I$ with

$$(g \circ f)'(x) = g'(f(x))\, f'(x).$$

Proof For each $x \in I$, since f is differentiable at x, the function

$$o(h) = f(x+h) - f(x) - f'(x)\, h \quad \text{defined for sufficiently small } h$$

satisfies that

$$\lim_{h \to 0} \frac{o(h)}{h} = 0.$$

Also since $f(I) \subseteq J$, the range of f is a subset of the domain of g, $y = f(x) \in J$ and since g is differentiable at $f(x)$, the function

$$o(k) = g(f(x)+k) - g(f(x)) - g'(f(x))\, k \quad \text{for sufficiently small } k$$

satisfies that

$$\lim_{k \to 0} \frac{o(k)}{k} = 0.$$

By setting $k = f(x+h) - f(x)$ and using that

$$\lim_{h \to 0} k = 0 \text{ by the continuity of } f \text{ at } x \text{ and} \lim_{h \to 0} \frac{k}{h} = f'(x),$$

we have

$$\begin{aligned}
& \lim_{h \to 0} \frac{g \circ f(x+h) - g \circ f(x) - g'(f(x))\, f'(x)\, h}{h} \\
= \;& \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x)) - g'(f(x))\; f'(x)\, h}{h} \\
= \;& \lim_{h \to 0} \frac{g(f(x) + f(x+h) - f(x)) - g(f(x)) - g'(f(x))\, \big[f(x+h) - f(x) - o(h)\big]}{h} \\
= \;& \lim_{h \to 0} \frac{g(f(x)+k) - g(f(x)) - g'(f(x))\, \big[k - o(h)\big]}{h} \\
= \;& \lim_{h \to 0} \frac{g(f(x)+k) - g(f(x)) - g'(f(x))\, k}{k}\; \frac{k}{h} + \lim_{h \to 0} \frac{o(h)}{h} \\
= \;& 0.
\end{aligned}$$

Hence $g \circ f$ is differentiable for all $x \in I$ with

$$(g \circ f)'(x) = g'(f(x))\, f'(x) \quad \text{for all } x \in I.$$